Gov 2001: Problem Set 6

Spring 2026

Instructions:

- The Problem set is due on **March 31, 11:59 PM Eastern Time**.
- Please upload a PDF of your solutions to Gradescope. Make sure to assign to each question all the pages with your work on that question.
- **Do not use AI assistants (ChatGPT, Claude, Copilot, etc.) on this problem set.** Work with each other instead. The struggle is where learning happens.
- Remember: 70% of your grade comes from in-class exams. Use problem sets to *learn*, not just to get answers.

Short Questions

1. Suppose $X_1, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\lambda)$. Write down the likelihood function $L(\lambda)$ and the log-likelihood function $\ell(\lambda)$.

Long Questions

2. Let X be a nonnegative random variable with $\mathbb{E}[X] = 10$ and $\text{Var}(X) = 36$.

   (a) Use **Markov's inequality** to give an upper bound for $\mathbb{P}(X \geq 25)$.

   (b) Use **Chebyshev's inequality** to give an upper bound for $\mathbb{P}(|X - 10| \geq 15)$.

   (c) Use part (b) to obtain an upper bound for $\mathbb{P}(X \geq 25)$ and compare it with part (a).

3. Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Pois}(\lambda)$ with $\lambda > 0$. Since $\mathbb{E}[X_i] = \text{Var}(X_i) = \lambda$, we can construct two plug-in estimators of λ:

$$\hat{\lambda}_1 = \bar{X}, \qquad \hat{\lambda}_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

   (a) Show that $\hat{\lambda}_2 = \overline{X^2} - (\bar{X})^2$.

   (b) Show that both estimators are consistent for λ.

   (c) Use CLT to derive the asymptotic distribution of $\sqrt{n}(\hat{\lambda}_1 - \lambda)$.

   (d) Use CLT and Delta method to derive the asymptotic distribution of $\sqrt{n}(\hat{\lambda}_2 - \lambda)$.

   (e) We already know that $\hat{\lambda}_2$ is biased and the bias-corrected estimator is $\hat{\lambda}_3 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{n}{n-1}\hat{\lambda}_2$. Does $\hat{\lambda}_3$ have the same asymptotic distribution with $\hat{\lambda}_2$?

4. Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, where σ^2 is known. The estimand is $\theta = \mu^2$.

(a) Write down the likelihood function and log-likelihood function for μ.

(b) Take the first-order condition and solve for the MLE $\hat{\mu}_{MLE}$ step by step.

(c) Define the MLE for θ as $\hat{\theta}_{MLE} = (\hat{\mu}_{MLE})^2$. Use CLT and Delta method to derive the asymptotic distribution of $\sqrt{n}(\hat{\theta}_{MLE} - \theta)$.

5. Let $Y_1, \ldots, Y_n$ be i.i.d. random variables such that $\mathbb{E}[Y_i^2] = 1$, $\text{Var}(Y_i^2) < \infty$, and define

$$S_n = \frac{1}{n}\sum_{i=1}^{n} Y_i^2, \qquad V_n = \frac{1}{n}\sum_{i=1}^{n}(Y_i^2 - S_n)^2.$$

Let's see how CLT, LLN, CMT, Slutsky, and Delta method are used in practice.

(a) What is the approximate distribution of S_n as $n \to \infty$? Your answer can include $\text{Var}(Y_i^2)$ but should not include $\mathbb{E}[Y_i^2]$.

(b) Show that $\sqrt{V_n} \xrightarrow{p} \sqrt{\text{Var}(Y_i^2)}$. (Hint: Use LLN and CMT.)

(c) Given parts (a) and (b), prove

$$\frac{\sqrt{n}(S_n - 1)}{\sqrt{V_n}} \xrightarrow{d} N(0, 1).$$

(Hint: Use Slutsky.)

(d) Find the constant c such that

$$c \cdot \frac{\sqrt{n}(\exp(S_n) - e)}{\sqrt{V_n}} \xrightarrow{d} N(0, 1).$$

(Hint: Use the Delta method.)

6. Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(p)$, where $p \in (0, 1)$, and let

$$\hat{p} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

We will construct confidence intervals for p using two different methods.

(a) Compute $\mathbb{E}[\hat{p}]$ and $\text{Var}(\hat{p})$.

(b) Use Chebyshev's inequality to construct a conservative $(1 - \alpha)$ confidence interval for p.

(c) Write the asymptotic distribution of $\sqrt{n}(\hat{p} - p)$ and $\hat{p}$.

(d) Let σ be the standard deviation of $\hat{p}$'s asymptotic distribution. Express σ in the form of p and construct $\hat{\sigma}$ by plugging in $\hat{p}$ for p.

(e) Use your result in (c) and (d) to construct an approximate $(1 - \alpha)$ confidence interval for p. Notice that your confidence interval shouldn't contain the unknown estimand p.