Working Session

Random Variables, Expectation, and Variance

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Gov 2001 · Harvard University

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Where We Are

Monday's foundation → today's practice

Monday: random variables, PMFs, PDFs, expected value, variance, Jensen's inequality.

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Today is a mix of *practice* and *new ideas*:

- **Two problems per concept** — one abstract (dice), one applied (courts, elections, wait times) — for both sides of the room
- Deeper work on **variance** and **Jensen** with worked examples
- New ideas: **independence**, joint events ($A \cap B$ = "A intersect B"), **monotonicity**
- **Indicator variables** and linearity of expectation as a shortcut

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- **Indicator variables** and linearity of expectation as a shortcut

Goal

Collect all these ideas, get your hands dirty, and walk out ready for the problem set.

Notation Reference

All the symbols in one place

Random Variables: The Basics

Symbol	Name	Meaning
X	Random variable	A function from outcomes to numbers
x	Realized value	A specific number X could equal
Ω	Sample space	Set of all possible outcomes
ω	Outcome	One element of Ω
$f(x)$	PMF or PDF	Probability (discrete) or density (continuous) at x
$\text{Supp}[X]$	Support	Values where $f(x) > 0$

Key distinction:

- X is **random** — we don't know its value yet
- x is a **number** — a specific value we're asking about

Distributions: PMF, PDF, CDF

Symbol	Name	Formula	Use
$f(x)$ or $p(x)$	PMF	$\mathbb{P}(X = x)$	Discrete: probability of exactly x
$f(x)$	PDF	—	Continuous: density at x
$F(x)$	CDF	$\mathbb{P}(X \leq x)$	Both: probability up to x

Key facts:

- PMF: $\sum_x f(x) = 1$ (probabilities sum to 1)
- PDF: $\int f(x)\,dx = 1$ (area under curve = 1)
- PDF can exceed 1! It's density, not probability.
- CDF: Always between 0 and 1, non-decreasing

Expectation and Variance

Concept	Discrete	Continuous
Expected value	$\mathbb{E}[X] = \sum_x x \cdot f(x)$	$\mathbb{E}[X] = \int x \cdot f(x)\, dx$
LOTUS	$\mathbb{E}[g(X)] = \sum_x g(x) \cdot f(x)$	$\mathbb{E}[g(X)] = \int g(x) \cdot f(x)\, dx$
Variance	$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$	

Key properties:

- Linearity: $\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$
- Variance: $\text{Var}[aX + b] = a^2\,\text{Var}[X]$ (constants disappear!)
- Jensen: If g is convex, $\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$

Worked Problems

Let's calculate together

Problem 1: PMF of a Die Roll

Let X = outcome of rolling a fair 6-sided die.

Questions:

1. What is the support of X?
2. Write out the PMF $f(x)$.
3. What is $\mathbb{P}(X \leq 3)$?
4. What is $\mathbb{P}(X > 4)$?

Problem 1: PMF of a Die Roll

Let X = outcome of rolling a fair 6-sided die.

Questions:

1. What is the support of X?
2. Write out the PMF $f(x)$.
3. What is $\mathbb{P}(X \leq 3)$?
4. What is $\mathbb{P}(X > 4)$?

Answers:

1. $\text{Supp}[X] = \{1, 2, 3, 4, 5, 6\}$
2. $f(x) = \frac{1}{6}$ for $x \in \{1, 2, 3, 4, 5, 6\}$, and $f(x) = 0$ otherwise
3. $\mathbb{P}(X \leq 3) = F(3) = \frac{3}{6} = \frac{1}{2}$
4. $\mathbb{P}(X > 4) = 1 - \mathbb{P}(X \leq 4) = 1 - \frac{4}{6} = \frac{1}{3}$

Our Running Example: A Redistricting Case

Setting up the court

A state supreme court with 5 justices hears a redistricting case. An analyst estimates each justice's probability of voting to strike down the map:

Justice	1	2	3	4	5
$\mathbb{P}$(strike)	0.70	0.60	0.55	0.40	0.35
$\mathbb{P}$(uphold)	0.30	0.40	0.45	0.60	0.65

Each justice's column sums to 1 — but the strike probabilities *across* justices don't need to.

Let X = total number of justices who vote to strike. If they vote **independently**, how do we get the PMF of X from these individual probabilities?

From Individual Probabilities to a PMF

$f(0)$: All justices uphold

Justice	1	2	3	4	5
$\mathbb{P}$(strike)	0.70	0.60	0.55	0.40	0.35
$\mathbb{P}$(uphold)	**0.30**	**0.40**	**0.45**	**0.60**	**0.65**

$\mathbb{P}(X = 0)$: *all* 5 justices uphold. Independence means we **multiply**:

$$f(0) = 0.30 \times 0.40 \times 0.45 \times 0.60 \times 0.65 = 0.02$$

From Individual Probabilities to a PMF

$f(5)$: All justices strike

Justice	1	2	3	4	5
$\mathbb{P}(\text{strike})$	**0.70**	**0.60**	**0.55**	**0.40**	**0.35**
$\mathbb{P}(\text{uphold})$	0.30	0.40	0.45	0.60	0.65

$\mathbb{P}(X = 5)$: *all* 5 strike. Same logic:

$$f(5) = 0.70 \times 0.60 \times 0.55 \times 0.40 \times 0.35 = 0.03$$

The middle values are harder — we have to sum over *which* justices strike.

From Individual Probabilities to a PMF

$f(1)$: Exactly one justice strikes — Justice 1

Justice	1	2	3	4	5
$\mathbb{P}$(strike)	**0.70**	0.60	0.55	0.40	0.35
$\mathbb{P}$(uphold)	0.30	**0.40**	**0.45**	**0.60**	**0.65**

For $f(1)$, exactly one justice strikes. There are 5 ways. Start with Justice 1:

$$\text{J1 strikes:} \quad 0.70 \times 0.40 \times 0.45 \times 0.60 \times 0.65 = 0.0491$$

From Individual Probabilities to a PMF

$f(1)$: Exactly one justice strikes — Justice 2

Justice	1	2	3	4	5
$\mathbb{P}$(strike)	0.70	**0.60**	0.55	0.40	0.35
$\mathbb{P}$(uphold)	**0.30**	0.40	**0.45**	**0.60**	**0.65**

Now Justice 2 is the lone striker:

J1 strikes: $0.70 \times 0.40 \times 0.45 \times 0.60 \times 0.65 = 0.0491$

J2 strikes: $0.30 \times 0.60 \times 0.45 \times 0.60 \times 0.65 = 0.0316$

From Individual Probabilities to a PMF

$f(1)$: Exactly one justice strikes — Justice 3

Justice	1	2	3	4	5
$\mathbb{P}$(strike)	0.70	0.60	**0.55**	0.40	0.35
$\mathbb{P}$(uphold)	**0.30**	**0.40**	0.45	**0.60**	**0.65**

Justice 3 is the lone striker:

$$\begin{aligned}
\text{J1 strikes:} \quad & 0.70 \times 0.40 \times 0.45 \times 0.60 \times 0.65 = 0.0491 \\
\text{J2 strikes:} \quad & 0.30 \times 0.60 \times 0.45 \times 0.60 \times 0.65 = 0.0316 \\
\text{J3 strikes:} \quad & 0.30 \times 0.40 \times 0.55 \times 0.60 \times 0.65 = 0.0257
\end{aligned}$$

From Individual Probabilities to a PMF

$f(1)$: Exactly one justice strikes — Justice 4

Justice	1	2	3	4	5
$\mathbb{P}$(strike)	0.70	0.60	0.55	**0.40**	0.35
$\mathbb{P}$(uphold)	**0.30**	**0.40**	**0.45**	0.60	**0.65**

Justice 4 is the lone striker:

J1 strikes: $0.70 \times 0.40 \times 0.45 \times 0.60 \times 0.65 = 0.0491$

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J4 strikes: $0.30 \times 0.40 \times 0.45 \times 0.40 \times 0.65 = 0.0140$

From Individual Probabilities to a PMF

$f(1)$: Exactly one justice strikes — Justice 5

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Justice 5 is the lone striker:

$$\begin{aligned}
\text{J1 strikes:}\quad & 0.70 \times 0.40 \times 0.45 \times 0.60 \times 0.65 = 0.0491 \\
\text{J2 strikes:}\quad & 0.30 \times 0.60 \times 0.45 \times 0.60 \times 0.65 = 0.0316 \\
\text{J3 strikes:}\quad & 0.30 \times 0.40 \times 0.55 \times 0.60 \times 0.65 = 0.0257 \\
\text{J4 strikes:}\quad & 0.30 \times 0.40 \times 0.45 \times 0.40 \times 0.65 = 0.0140 \\
\text{J5 strikes:}\quad & 0.30 \times 0.40 \times 0.45 \times 0.60 \times 0.35 = 0.0113
\end{aligned}$$

From Individual Probabilities to a PMF

Summing the five cases and the full PMF

Justice	1	2	3	4	5
$\mathbb{P}$(strike)	0.70	0.60	0.55	0.40	0.35
$\mathbb{P}$(uphold)	0.30	0.40	0.45	0.60	0.65

$$f(1) = 0.0491 + 0.0316 + 0.0257 + 0.0140 + 0.0113 = \boxed{0.13}$$

For $f(2)$: $\binom{5}{2} = 10$ terms. For $f(3)$: another 10. Same logic, more adding. The full PMF:

x	0	1	2	3	4	5
$f(x)$	0.02	0.13	0.31	0.34	0.17	0.03

Why Exactly 5 Terms?

A quick note on counting before we move on

You just saw us write out 5 separate products for $f(1)$. Why 5?

Because there are exactly 5 ways to choose which *one* justice strikes. Mathematicians write this as:

$$\binom{5}{1} = \frac{5!}{1! \cdot 4!} = 5$$

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And that's why $f(2)$ would require $\binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$ terms — 10 ways to pick which *two* justices strike.

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And that's why $f(2)$ would require $\binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$ terms — 10 ways to pick which *two* justices strike.

Key point: if all 5 justices had the *same* strike probability p, those terms would all be equal and you could use the **binomial** shortcut: $f(k) = \binom{5}{k} p^k (1-p)^{5-k}$.

But our justices have *different* probabilities — so we're stuck adding up each term individually. That's exactly what we just did.

Problem 1b: A State Supreme Court Vote

Recall the redistricting court. X = number who vote to strike down the map:

x	0	1	2	3	4	5
$f(x)$	0.02	0.13	0.31	0.34	0.17	0.03

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Questions:

1. Verify this is a valid PMF.
2. What is $\mathbb{P}(X \geq 3)$? (i.e., the map gets struck down)
3. What is $\mathbb{P}(X = 2 \text{ or } X = 3)$?

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Answers:

1. $\sum f(x) = 0.02 + 0.13 + 0.31 + 0.34 + 0.17 + 0.03 = 1 \checkmark$, all $f(x) \geq 0 \checkmark$
2. $\mathbb{P}(X \geq 3) = 0.34 + 0.17 + 0.03 = \boxed{0.54}$
3. $\mathbb{P}(X = 2 \text{ or } X = 3) = 0.31 + 0.34 = 0.65$

Problem 2: Expected Value of Die Roll

Same die roll: $X \in \{1, 2, 3, 4, 5, 6\}$, each with probability $\frac{1}{6}$.

Calculate $\mathbb{E}[X]$.

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Calculate $\mathbb{E}[X]$.

Solution:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x=1}^{6} x \cdot f(x) = \sum_{x=1}^{6} x \cdot \frac{1}{6} \\
&= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) \\
&= \frac{1}{6}(21) = \boxed{3.5}
\end{aligned}$$

Note: $\mathbb{E}[X] = 3.5$ is not in the support! The mean doesn't have to be a possible value.

Problem 2b: Expected Votes to Strike Down

Same redistricting case. Recall the PMF:

x	0	1	2	3	4	5
$f(x)$	0.02	0.13	0.31	0.34	0.17	0.03

Calculate $\mathbb{E}[X]$.

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Calculate $\mathbb{E}[X]$.

Solution:

$$\begin{aligned}\mathbb{E}[X] &= 0(0.02) + 1(0.13) + 2(0.31) + 3(0.34) + 4(0.17) + 5(0.03) \\ &= 0 + 0.13 + 0.62 + 1.02 + 0.68 + 0.15 = \boxed{2.60}\end{aligned}$$

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On average, about 2.6 justices vote to strike — just short of the 3-vote majority. The map survives more often than not, but barely.

Problem 3: Variance of Die Roll

We know $\mathbb{E}[X] = 3.5$. Now find $\text{Var}[X]$.

Step 1: Calculate $\mathbb{E}[X^2]$.

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$$
\begin{aligned}
\mathbb{E}[X^2] &= \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) \\
&= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6} \approx 15.17
\end{aligned}
$$

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Step 2: Apply the formula.

$$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{91}{6} - (3.5)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12} \approx 2.92$$

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Jensen check: $\mathbb{E}[X^2] = 15.17 > (\mathbb{E}[X])^2 = 12.25$ ✓

Problem 3b: Variance of Court Vote

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$$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 7.90 - (2.60)^2 = 7.90 - 6.76 = \boxed{1.14}$$

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Jensen check: $\mathbb{E}[X^2] = 7.90 > (\mathbb{E}[X])^2 = 6.76$ ✓

Compare: the die had Var = 2.92 over 6 values. The court vote has Var = 1.14 — much less spread because the PMF concentrates around 2–3.

Problem 4: Indicator Variables

The bridge between probability and expectation

An **indicator variable** D equals 1 if an event occurs, 0 otherwise.

Key insight:

$$\mathbb{E}[D] = 1 \cdot \mathbb{P}(D = 1) + 0 \cdot \mathbb{P}(D = 0) = \mathbb{P}(D = 1)$$

The Bridge

Expected value of an indicator = Probability of the event

Example: Draw one card from a deck. Let $D = 1$ if it's an Ace.

$$\mathbb{E}[D] = \mathbb{P}(\text{Ace}) = \frac{4}{52} = \frac{1}{13}$$

Problem 4b: Indicators on the Court

Same court, easier path to $\mathbb{E}[X]$

We already know the individual strike probabilities. Define $D_i = 1$ if Justice i votes to strike, so $X = D_1 + D_2 + D_3 + D_4 + D_5$.

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By linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[D_1] + \mathbb{E}[D_2] + \mathbb{E}[D_3] + \mathbb{E}[D_4] + \mathbb{E}[D_5] = 0.70 + 0.60 + 0.55 + 0.40 + 0.35 = 2.60$$

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Same answer as Problem 2b — **it has to be**. It's the same X.

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Same answer as Problem 2b — **it has to be**. It's the same X.

But think about what we *didn't* need: we never touched the PMF table. No summing $x \cdot f(x)$ over all 6 outcomes. Linearity gave us $\mathbb{E}[X]$ straight from the individual probabilities.

Problem 5: Counting Aces in a Hand

Indicator variables + Linearity of expectation

Deal 5 cards from a standard deck. Let X = number of Aces.

The hard way: Find the full PMF of X, then compute $\mathbb{E}[X]$.

The easy way: Use indicator variables!

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The easy way: Use indicator variables!

Define: $D_i = 1$ if card i is an Ace, for $i = 1, 2, 3, 4, 5$.

Then: $X = D_1 + D_2 + D_3 + D_4 + D_5$

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Then: $X = D_1 + D_2 + D_3 + D_4 + D_5$

By linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[D_1] + \mathbb{E}[D_2] + \mathbb{E}[D_3] + \mathbb{E}[D_4] + \mathbb{E}[D_5]$$

Each $\mathbb{E}[D_i] = \mathbb{P}(\text{card } i \text{ is Ace}) = \frac{4}{52} = \frac{1}{13}$

$$\mathbb{E}[X] = 5 \times \frac{1}{13} = \boxed{\frac{5}{13} \approx 0.385}$$

Problem 5: Why This Works

The magic of linearity

Wait — aren't the cards dependent? After drawing an Ace, fewer Aces remain!

Yes, but it doesn't matter for expectation!

Linearity of expectation says:

$$\mathbb{E}[D_1 + D_2 + \cdots + D_n] = \mathbb{E}[D_1] + \mathbb{E}[D_2] + \cdots + \mathbb{E}[D_n]$$

always — regardless of whether the D_i are independent.

Key Takeaway

To find $\mathbb{E}[\text{count}]$, just sum the individual probabilities.
No need to find the joint distribution!

This is why indicator variables are the "fundamental bridge" between probability and expectation.

Problem 5b: How Many Swing States?

Linearity of expectation in elections

A candidate contests 7 battleground states. Let X = number won.

Define $D_i = 1$ if the candidate wins state i, where $\mathbb{P}(D_i = 1)$ comes from a forecasting model:

State	PA	MI	WI	AZ	GA	NV	NC
$\mathbb{P}(D_i = 1)$	0.55	0.52	0.53	0.45	0.42	0.50	0.38

Problem 5b: How Many Swing States?

Linearity of expectation in elections

A candidate contests 7 battleground states. Let X = number won.

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These outcomes are **highly dependent** (national mood affects all states).

Problem 5b: How Many Swing States?

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$\mathbb{P}(D_i = 1)$	0.55	0.52	0.53	0.45	0.42	0.50	0.38

These outcomes are **highly dependent** (national mood affects all states).

But linearity doesn't care!

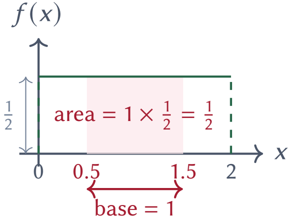
$$\mathbb{E}[X] = 0.55 + 0.52 + 0.53 + 0.45 + 0.42 + 0.50 + 0.38 = 3.35$$

On average, the candidate wins about 3.4 of 7 battlegrounds — but the dependence means the actual outcome is much more "all or nothing" than 3.35 suggests.

The Uniform Distribution: Quick Review

Density is a height, probability is an area

If $X \sim \text{Uniform}(0, 2)$, the PDF is $f(x) = \frac{1}{2}$ for $x \in [0, 2]$:



Interpreting density: $f(x) = \frac{1}{2}$ is *not* a probability — it's a *height*. To get probability, take the **area under the curve**:

$$\mathbb{P}(0.5 \leq X \leq 1.5) = \underbrace{(1.5 - 0.5)}_{\text{base}} \times \underbrace{\frac{1}{2}}_{\text{height}} = \frac{1}{2}$$

The Uniform Distribution: Key Formulas

Reference for Problems 6–7

In general, for $X \sim \text{Uniform}(a, b)$:

$$f(x) = \frac{1}{b-a} \quad \text{for } x \in [a,\ b]$$

Probability (area of a rectangle):

$$\mathbb{P}(c \leq X \leq d) = (d-c) \cdot \frac{1}{b-a}$$

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Key formulas:

Expected value	$\mathbb{E}[X] = \frac{a+b}{2}$	(the midpoint)
Variance	$\text{Var}[X] = \frac{(b-a)^2}{12}$	(wider support $\Rightarrow$ more spread)

Problem 6: Uniform Distribution

Let $X \sim \text{Uniform}(0, 2)$. The PDF is $f(x) = \frac{1}{2}$ for $x \in [0, 2]$.

Questions:

1. What is $\mathbb{P}(X \leq 1)$?
2. What is $\mathbb{P}(0.5 < X < 1.5)$?
3. What is $\mathbb{E}[X]$?

Problem 6: Uniform Distribution

Let $X \sim \text{Uniform}(0, 2)$. The PDF is $f(x) = \frac{1}{2}$ for $x \in [0, 2]$.

Questions:

1. What is $\mathbb{P}(X \leq 1)$?
2. What is $\mathbb{P}(0.5 < X < 1.5)$?
3. What is $\mathbb{E}[X]$?

Answers:

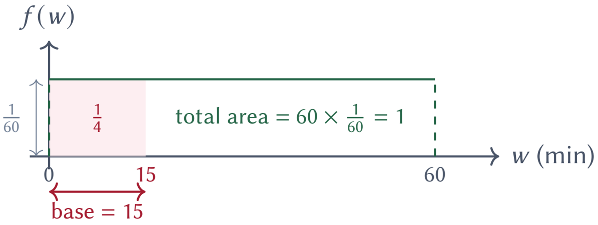
1. $\mathbb{P}(X \leq 1) = \int_0^1 \frac{1}{2}\, dx = \frac{1}{2}$
2. $\mathbb{P}(0.5 < X < 1.5) = \int_{0.5}^{1.5} \frac{1}{2}\, dx = \frac{1}{2}(1.5 - 0.5) = \frac{1}{2}$
3. $\mathbb{E}[X] = \int_0^2 x \cdot \frac{1}{2}\, dx = \frac{1}{2} \cdot \frac{x^2}{2}\Big|_0^2 = \frac{1}{2} \cdot 2 = 1$

For $\text{Uniform}(a, b)$: $\mathbb{E}[X] = \frac{a+b}{2}$ (the midpoint)

Voter Wait Times: Visualizing the PDF

$W \sim \text{Uniform}(0, 60)$ minutes

A precinct's wait time follows Uniform$(0, 60)$. The PDF is $f(w) = \frac{1}{60}$ for $w \in [0, 60]$:



$$\mathbb{P}(W \le 15) = \underbrace{15}_{\text{base}} \times \underbrace{\frac{1}{60}}_{\text{height}} = \frac{1}{4}$$

Same logic as before — just a wider, shorter rectangle.

Problem 6b: Voter Wait Times

A precinct's wait time W follows Uniform$(0, 60)$ minutes. The PDF is $f(w) = \frac{1}{60}$ for $w \in [0, 60]$.

Questions:

1. What is $\mathbb{P}(W \leq 15)$? (voter waits 15 min or less)
2. What is $\mathbb{P}(W > 45)$? (voter waits more than 45 min)
3. What is $\mathbb{E}[W]$?

Problem 6b: Voter Wait Times

A precinct's wait time W follows Uniform$(0, 60)$ minutes. The PDF is $f(w) = \frac{1}{60}$ for $w \in [0, 60]$.

Questions:

1. What is $\mathbb{P}(W \leq 15)$? (voter waits 15 min or less)
2. What is $\mathbb{P}(W > 45)$? (voter waits more than 45 min)
3. What is $\mathbb{E}[W]$?

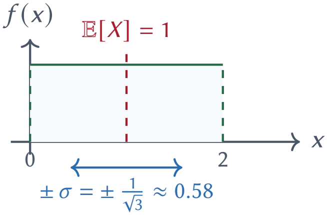
Answers:

1. $\mathbb{P}(W \leq 15) = \int_0^{15} \frac{1}{60}\, dw = \frac{15}{60} = \frac{1}{4}$
2. $\mathbb{P}(W > 45) = 1 - \mathbb{P}(W \leq 45) = 1 - \frac{45}{60} = \frac{1}{4}$
3. $\mathbb{E}[W] = \frac{0+60}{2} =$ 30 minutes

Problem 7: Variance of Uniform

How spread out is X?

Still $X \sim \text{Uniform}(0, 2)$. We know $\mathbb{E}[X] = 1$. How spread out are values around the mean?



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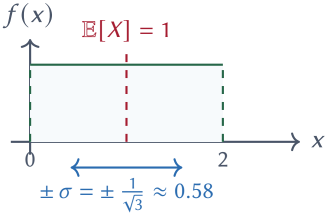

Step 1: $\mathbb{E}[X^2] = \int_0^2 x^2 \cdot \frac{1}{2}\, dx = \frac{1}{2} \cdot \frac{x^3}{3}\Big|_0^2 = \frac{4}{3}$

Step 2: $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{4}{3} - 1 = \frac{1}{3}$, so $\sigma = \frac{1}{\sqrt{3}} \approx 0.58$

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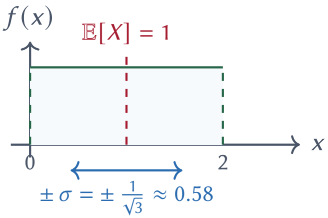

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General formula: For $\text{Uniform}(a, b)$, $\text{Var}[X] = \frac{(b-a)^2}{12}$. Here: $\frac{4}{12} = \frac{1}{3}$ ✓

Problem 7b: Variance of Voter Wait Times

Still $W \sim \text{Uniform}(0, 60)$. We know $\mathbb{E}[W] = 30$. Find $\text{Var}[W]$.

Step 1: Calculate $\mathbb{E}[W^2]$.

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Step 2: Apply the formula.

$$\text{Var}[W] = \mathbb{E}[W^2] - (\mathbb{E}[W])^2 = 1200 - 900 = \boxed{300}$$

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$$\text{Var}[W] = \mathbb{E}[W^2] - (\mathbb{E}[W])^2 = 1200 - 900 = \boxed{300}$$

Standard deviation: $\sigma = \sqrt{300} \approx 17.3$ minutes.

Check: $\text{Uniform}(a, b)$ formula gives $\frac{(60-0)^2}{12} = \frac{3600}{12} = 300$ ✓

Problem 8: Is $g(x) = x^2$ Convex?

Setting up Jensen's inequality

Jensen's inequality requires g to be **convex**. How do we check?

Second Derivative Test

A twice-differentiable function g is convex if and only if $g''(x) \geq 0$ for all x.

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Setting up Jensen's inequality

Jensen's inequality requires g to be **convex**. How do we check?

Second Derivative Test

A twice-differentiable function g is convex if and only if $g''(x) \geq 0$ for all x.

Check: Let $g(x) = x^2$.

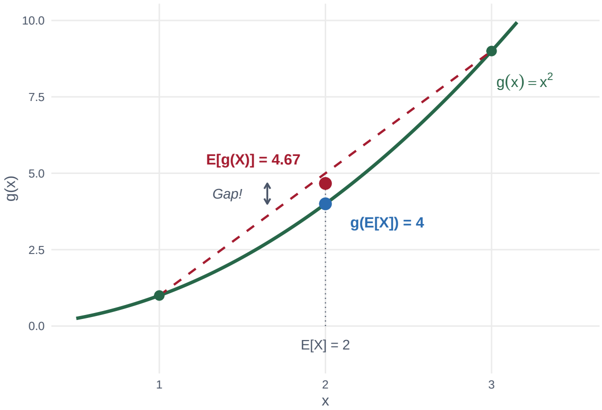
$$g'(x) = 2x$$
$$g''(x) = 2 > 0 \quad \text{for all } x \checkmark$$

So $g(x) = x^2$ is convex, and Jensen tells us:

$$\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$$

Visualizing Jensen's Inequality

Why the chord lies above the curve



For convex functions, the average of $g(X)$ (on the chord) always exceeds g evaluated at the average (on the curve).

Problem 9: Jensen's Inequality in Action

Suppose X takes values 1, 2, 3 with equal probability $\frac{1}{3}$ each.

Verify Jensen's inequality for $g(x) = x^2$ (convex, since $g'' = 2 > 0$).

Problem 9: Jensen's Inequality in Action

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$$\mathbb{E}[X^2] = \frac{1}{3}(1^2 + 2^2 + 3^2) = \frac{1}{3}(1 + 4 + 9) = \frac{14}{3} \approx 4.67$$

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Jensen says: $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$

Check: $4.67 \geq 4$ ✓

Variance: $\text{Var}[X] = 4.67 - 4 = \frac{2}{3} \geq 0$ ✓

Problem 9b: Jensen and Campaign Advertising

A campaign's vote share gain from ad spending X (in \$millions) is $g(X) = \sqrt{X}$ – **concave** (diminishing returns), since $g''(x) = -\frac{1}{4}x^{-3/2} < 0$.

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Right side: $\sqrt{\mathbb{E}[X]} = \sqrt{5} \approx 2.24$

Jensen (concave): $\mathbb{E}[\sqrt{X}] = 2.0 \leq \sqrt{\mathbb{E}[X]} = 2.24$ ✓

Political Implication

Uncertain ad budgets perform *worse* on average than spending the mean amount for certain. Volatility hurts when returns are concave.

Two More Properties

Monotonicity and Independence

Monotonicity of Expectation

Bigger inputs, bigger expected values

Monotonicity

If $X \leq Y$ for all outcomes (i.e., $X(\omega) \leq Y(\omega)$ for all ω), then:

$$\mathbb{E}[X] \leq \mathbb{E}[Y]$$

Intuition: If X never exceeds Y, its average can't exceed Y's average either.

Example: Suppose you're comparing two jobs:

- Job A pays \$40k, \$50k, or \$60k (equally likely)
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Every outcome of B exceeds A, so:

$$\mathbb{E}[A] = 50\text{k} \leq \mathbb{E}[B] = 55\text{k}\checkmark$$

Problem 10: Monotonicity in Action

Let X be a random variable with $0 \leq X \leq 1$ (always).

Questions:

1. What can you say about $\mathbb{E}[X]$?
2. Since $0 \leq X \leq 1$, we know $X^2 \leq X$ (always). What does monotonicity tell us about $\mathbb{E}[X^2]$ vs $\mathbb{E}[X]$?

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Answers:

1. By monotonicity: $\mathbb{E}[0] \leq \mathbb{E}[X] \leq \mathbb{E}[1]$, so $0 \leq \mathbb{E}[X] \leq 1$.
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2. Since $X^2 \leq X$ always, monotonicity gives $\mathbb{E}[X^2] \leq \mathbb{E}[X]$.

Check with Uniform$(0,1)$: $\mathbb{E}[X^2] = \frac{1}{3}$ and $\mathbb{E}[X] = \frac{1}{2}$. Indeed $\frac{1}{3} \leq \frac{1}{2}$ ✓

Problem 10b: Rain and Voter Turnout

Let T_R = precinct turnout on a rainy day, T_C = turnout on a clear day.

Research shows rain depresses turnout: for every possible election scenario, $T_R(\omega) \leq T_C(\omega)$.

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Questions:

1. What does monotonicity tell us?
2. If $\mathbb{E}[T_C] = 0.62$, what can you say about $\mathbb{E}[T_R]$?

Answers:

1. Since $T_R \leq T_C$ always, monotonicity gives $\mathbb{E}[T_R] \leq \mathbb{E}[T_C]$.
2. $\mathbb{E}[T_R] \leq 0.62$. (We can bound the mean without knowing the full distribution of T_R.)

Monotonicity is a simple but powerful bounding tool — it lets you make statements about expectations from qualitative ordering alone.

Independence of Random Variables

Quick review

Independence

X and Y are **independent** ($X \perp\!\!\!\perp Y$) if knowing the value of one tells you nothing about the other:

$$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y) \quad \text{for all } x, y$$

Key consequences when $X \perp\!\!\!\perp Y$:

- $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$
- $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$
- $\text{Cov}[X, Y] = 0$

Warning: $\text{Cov}[X, Y] = 0$ does NOT imply independence! (We'll see why in Week 4.)

Problem 11: Independence and Variance

Roll two fair dice independently. Let X = first die, Y = second die, $S = X + Y$.

Questions:

1. Find $\mathbb{E}[S]$.
2. Find $\text{Var}[S]$.

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1. Find $\mathbb{E}[S]$.
2. Find $\text{Var}[S]$.

Part 1: By linearity (always works):

$$\mathbb{E}[S] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3.5 + 3.5 = \boxed{7}$$

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Part 2: Since $X \perp\!\!\!\perp Y$, variances add:

$$\text{Var}[S] = \text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] = \frac{35}{12} + \frac{35}{12} = \boxed{\tfrac{35}{6} \approx 5.83}$$

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Note: $\mathbb{E}[S]$ uses linearity (works always). $\mathrm{Var}[S] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ uses **independence**. Without independence, we'd need $\mathrm{Cov}[X, Y]$ too.

Problem 11b: Combining Independent Precincts

Two precincts report independently. Let V_A = votes for a candidate in Precinct A, V_B = votes in Precinct B. Total: $T = V_A + V_B$.

Given: $\mathbb{E}[V_A] = 800$, $\text{Var}[V_A] = 2500$, $\mathbb{E}[V_B] = 1200$, $\text{Var}[V_B] = 4000$.

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$$\mathbb{E}[T] = \mathbb{E}[V_A] + \mathbb{E}[V_B] = 800 + 1200 = 2000$$

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Part 1: By linearity (always works):

$$\mathbb{E}[T] = \mathbb{E}[V_A] + \mathbb{E}[V_B] = 800 + 1200 = 2000$$

Part 2: Since $V_A \perp\!\!\!\perp V_B$, variances add:

$$\text{Var}[T] = \text{Var}[V_A] + \text{Var}[V_B] = 2500 + 4000 = 6500$$

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$$\mathbb{E}[T] = \mathbb{E}[V_A] + \mathbb{E}[V_B] = 800 + 1200 = \boxed{2000}$$

Part 2: Since $V_A \perp\!\!\!\perp V_B$, variances add:

$$\text{Var}[T] = \text{Var}[V_A] + \text{Var}[V_B] = 2500 + 4000 = \boxed{6500}$$

What if the precincts were **not** independent — say, both affected by local weather? Then variances don't simply add. We'll need a new tool called **covariance** to handle that case. Coming soon.

Summary

What we practiced today

- **PMF/PDF**: Writing out distributions, computing probabilities
- **Expected value**: The weighted average formula
- **Variance**: $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- **Indicator variables**: $\mathbb{E}[D] = \mathbb{P}(\text{event})$
- **Linearity**: $\mathbb{E}[\sum D_i] = \sum \mathbb{E}[D_i]$ — works even with dependence!
- **Jensen**: $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$, so variance ≥ 0
- **Monotonicity**: $X \leq Y$ always $\implies \mathbb{E}[X] \leq \mathbb{E}[Y]$
- **Independence**: $X \perp\!\!\!\perp Y \implies \text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$

For Your Problem Sets

Practice these calculations until they're automatic.
The notation should feel natural, not foreign.

Next Time

Week 3: Famous Distributions

- Bernoulli and Binomial
- Uniform (discrete and continuous)
- Normal — the star of the show
- Poisson — for counts

These distributions will appear throughout the course. *Master their properties now.*