Worksheet 2: The Wald Estimator Equals 2SLS

Gov 51 · Scott Cunningham

Spring 2026

Worksheet 1 showed $\beta_1 = \text{Cov}(Y, Z)/\text{Cov}(D, Z)$. This worksheet shows that running two OLS regressions (2SLS) gives the same answer. We need two additional properties about variance.

Two additional properties we will use

P4. Variance of a scaled variable: $\text{Var}(aX) = a^2\,\text{Var}(X)$

P5. OLS slope formula: $\hat{\beta} = \text{Cov}(Y, X) \;/\; \text{Var}(X)$

Also recall P1–P3 from Worksheet 1 (linearity, constants, independence).

Setup. Two-stage least squares runs two regressions.

Stage 1 (first stage): regress D on Z. By P5, the slope is:

$$\hat{\pi}_1 \;=\; \frac{\text{Cov}(D,\, Z)}{\text{Var}(Z)}$$

Save the fitted values: $\hat{D}_i = \hat{\pi}_0 + \hat{\pi}_1 Z_i$.

Stage 2 (second stage): regress Y on $\hat{D}$. By P5, the 2SLS estimate is:

$$\hat{\beta}_1^{\text{2SLS}} \;=\; \frac{\text{Cov}(Y,\, \hat{D})}{\text{Var}(\hat{D})}$$

Goal: show this equals $\text{Cov}(Y, Z)/\text{Cov}(D, Z)$.

Step 1. Substitute $\hat{D} = \hat{\pi}_0 + \hat{\pi}_1 Z$ into the numerator.

$$\text{Cov}(Y,\, \hat{D}) \;=\; \text{Cov}(Y,\; \hat{\pi}_0 + \hat{\pi}_1 Z)$$

We replace $\hat{D}$ with what it actually is: the fitted values from stage 1.

Step 2. Apply P1 (linearity) and P2 (constants) to simplify.

$$\text{Cov}(Y,\ \hat{\pi}_0 + \hat{\pi}_1 Z) \;=\; \underbrace{\text{Cov}(Y,\ \hat{\pi}_0)}_{=0} + \hat{\pi}_1\,\text{Cov}(Y,\ Z) \;=\; \hat{\pi}_1\,\text{Cov}(Y,\ Z)$$

The constant $\hat{\pi}_0$ drops out (P2). We pull $\hat{\pi}_1$ out front (P1).

Step 3. Substitute $\hat{D} = \hat{\pi}_0 + \hat{\pi}_1 Z$ into the denominator. Apply P4 (constants don't affect variance; $\text{Var}(\hat{\pi}_1 Z) = \hat{\pi}_1^2\,\text{Var}(Z)$):

$$\text{Var}(\hat{D}) \;=\; \text{Var}(\hat{\pi}_0 + \hat{\pi}_1 Z) \;=\; \text{Var}(\hat{\pi}_1 Z) \;=\; \hat{\pi}_1^2\,\text{Var}(Z)$$

Adding a constant $\hat{\pi}_0$ does not change the variance. Then we apply P4.

Step 4. Form the ratio using the results from Steps 2 and 3.

$$\hat{\beta}_1^{\text{2SLS}} \;=\; \frac{\hat{\pi}_1\,\text{Cov}(Y,Z)}{\hat{\pi}_1^2\,\text{Var}(Z)} \;=\; \frac{\text{Cov}(Y,Z)}{\hat{\pi}_1\,\text{Var}(Z)}$$

One factor of $\hat{\pi}_1$ cancels. We can do this because $\hat{\pi}_1 \neq 0$ (relevance assumption).

Step 5. Substitute $\hat{\pi}_1 = \text{Cov}(D,Z)/\text{Var}(Z)$ from the stage-1 formula (P5).

$$\frac{\text{Cov}(Y,Z)}{\hat{\pi}_1\,\text{Var}(Z)} \;=\; \frac{\text{Cov}(Y,Z)}{\frac{\text{Cov}(D,Z)}{\text{Var}(Z)}\cdot\text{Var}(Z)} \;=\; \frac{\text{Cov}(Y,Z)}{\text{Cov}(D,Z)}$$

$\text{Var}(Z)$ cancels in the denominator. We are left with exactly the Wald estimator.

$$\hat{\beta}_1^{\text{2SLS}} \;=\; \frac{\text{Cov}(Y,Z)}{\text{Cov}(D,Z)} \;=\; \hat{\beta}_1^{\text{Wald}}$$

Two-stage least squares and the Wald estimator are the same thing. 2SLS is just the computational procedure for arriving at the Wald ratio.